Model structures, categorial quotients and representations of super commutative Hopf algebras II The case Gl(m|n)

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1 Introduction

Let $k^{m|n}$ be a $\mathbb{Z}/2$-graded vector space $X = k^m \oplus k^n$ over a field k of characteristic zero with even part k^m and odd part k^n. The super linear group $G = Gl(m|n)$ contains the classical reductive algebraic k-group $Gl(m) \times Gl(n)$. The Lie super algebra of G is $Lie(G) = End_k(k^m \oplus k^n)$, which decomposes into the subspace of endomorphisms which preserve respectively do not preserve the $\mathbb{Z}/2\mathbb{Z}$-grading. The even part of $Lie(G)$ can be identified with $Lie(Gl(m) \times Gl(n))$. The Lie super bracket on $End_k(k^m \oplus k^n)$ is defined by $[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X$ for graded endomorphisms X, Y in $End_k(k^m \oplus k^n)$. We suppose $m \geq n$.

An algebraic representation of G over k is a homomorphism

$$\rho : Gl(m) \times Gl(n) \longrightarrow Gl(V)$$

of algebraic groups over k, where $V = V_+ \oplus V_-$ is a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded k-vectorspace, together with a k-linear map

$$Lie(\rho) : Lie(G) \longrightarrow End(V)$$

so that

1. The parity on V is defined by the eigenspaces of $\rho(E_m, -E_n)$,

2. $Lie(\rho)$ is parity preserving for the natural $\mathbb{Z}/2\mathbb{Z}$-grading on $End_k(V)$ induced by $V = V_+ \oplus V_-$,

3. $Lie(\rho)$ is ρ-equivariant and coincides with the Lie derivative of ρ on the even part $Lie(Gl(m) \times Gl(n))$ of $Lie(G)$,

4. $Lie(\rho)$ respects the Lie super bracket.

A one dimensional representation ρ of G, which is the analog of the determinant, is the Berezin $Ber_{m|n}$ defined by $\rho(g_1 \times g_2) = det(g_1)/det(g_2)$ so that $Lie(\rho)$ is the super trace on $End(k^{m|n})$. The representation space of the Berezin is $k^{1|0}$ or $k^{0|1}$ depending on n modulo two.

Let $\mathcal{T}$ denote the abelian k-linear tensor category of algebraic representations of G. As a k-linear abelian category $\mathcal{T}$ decomposes into a direct sum of blocks Λ. We show that there exists a purely transcendent field extension K/k of transcendency degree n and a K-linear weakly exact tensor functor (see section 17)

$$\varphi : \mathcal{T} \otimes_k K \longrightarrow sRep_K(H) \quad , \quad H = Gl(m-n)$$

from the K-linear scalar extension of $\mathcal{T}$ to the semisimple category of finite dimensional algebraic super representations of the reductive algebraic K-group $H = Gl(m-n)$ defined over K. The simple objects of $sRep_K(H)$ are the $\rho[i]$ for the irreducible algebraic representations ρ of H, up to a parity shift of the grading ($i = 0$ or $i = 1$). We prove in corollary 4 that the image of a simple object V in $\mathcal{T}$ becomes zero under the functor φ if and only if V is a simple object, which is not maximal atypical (see section 2). On the other hand, in the main theorem of section 9 we prove that for maximal atypical simple objects the image $\varphi(V)$ is an isotypic representation $m(V) \cdot \rho(V)[p(V)]$ in $sRep_K(H)$, where the multiplicity $m(V)$ is > 0 and where $\rho(V)$ is an irreducible representation of H which only depends on the block Λ of V. The parity shift $p(V)$ can be easily computed. The computation of the multiplicity $m(V)$ is more subtle. We show

$$1 \leq m(V) \leq n! \,.$$

If V is simple with highest weight μ, we prove in section 14 the recursion formula

$$m(\mu) = \binom{n}{n_1, \ldots, n_r} \cdot \prod_{i=1}^{r} m(\mu_i)$$

for $m(V) = m(\mu)$, which allows to express $m(\mu)$ in terms of multinomials coefficients and multiplicies $m(\mu_i)$ for smaller n. Via

$$sdim_k(V) = (-1)^{p(V)} \cdot m(V) \cdot dim_k(\rho(V))$$

this formula for $m(\mu)$ gives a rather explicit formula for the super dimension of a maximal atypical irreducible representation V with weight μ, since the classical Weyl dimension formula for $Gl(m-n)$ computes $dim_k(\rho(V))$.

2 Weights

Let k be a field of characteristic zero, and let $G = Gl(m|n)$ denote the superlinear group over k. In the following we always assume $m \geq n$.

In this section we review some fundamental facts on highest weights. For more details see [BS1] and [BS4]. Let $\mathcal{T}$ denote the k-linear rigid tensor category $Rep_k(\mu, G)$ of k-linear algebraic super representations ρ of G on $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional k-super vector spaces, such that $\rho(id_m, -id_n)$ induces the parity automorphism of the underlying $\mathbb{Z}/2\mathbb{Z}$-grading of the representation space of ρ. Here $\mu : \mathbb{Z}/2\mathbb{Z} \mapsto (id_m, -id_n) \in Gl(m) \times Gl(n) \subset Gl(m|n)$ as in [BS4]. The category $\mathcal{T}$ admits a k-linear anti-involution $* : \mathcal{T} \to \mathcal{T}$ so that $A^* \cong A$ holds for all simple objects V and all simple projective objects V of $\mathcal{T}$. The intrinsic dimension $\chi(V)$ in a rigid tensor category $\mathcal{T}$ with $End_{\mathcal{T}}(\mathbf{1}) \cong k$ is $\chi(A) = eval_V \circ coeval_V$. It is preserved by tensor functors. In our case $\chi(V)$ is the super dimension $sdim_k(A) = dim_k(V_+) - dim_k(V_-)$ of the underlying super vectorspace $V = V_+ \oplus V_-$. This is easily seen using the forget functor $\mathcal{T} \to svec_k$ to the category $svec_k$ of finite dimensional k-super vector spaces.

The isomorphism classes X^+ of the irreducible finite dimensional representations of $Gl(m|n)$ are indexed by their highest weights $\lambda = (\lambda_1, ..\lambda_m; \lambda_{m+1}, .., \lambda_{m+n})$. Here $\lambda_1 \geq ... \geq \lambda_m$ and $\lambda_{m+1} \geq ... \geq \lambda_{m+n}$ are integers, and every $\lambda \in \mathbb{Z}^{m+n}$ satisfying these inequalities occurs as the highest weight of an irreducible representation $L(\lambda)$. The trivial representation $\mathbf{1}$ corresponds to $\lambda = 0$. Assigned to each highest weight $\lambda \in X^+$ are two subsets of the numberline $\mathbf{Z}$, namely the set

$$I_\times(\lambda) = \{\lambda_1, \lambda_2 - 1, ..., \lambda_m - m + 1\}$$

of cardinality m, respectively the set of cardinality n

$$I_\circ(\lambda) = \{1 - m - \lambda_{m+1}, 2 - m - \lambda_{m+2}, ..., n - m - \lambda_{m+n}\} .$$

Following the notations of [BS4] the integers in $I_\times(\lambda) \cap I_\circ(\lambda)$ are labeled by $\vee$, the remaining ones in $I_\times(\lambda)$ resp. $I_\circ(\lambda)$ are labeled by $\times$ resp. $\circ$. All other integers are then labeled by $\wedge$. This labeling of the numberline $\mathbb{Z}$ uniquely characterizes the weight vector λ. If the label $\vee$ occurs r times in the labeling, then r is called the degree of atypicality of λ. Notice that $0 \leq r \leq n$, and λ is called maximal atypical if $r = n$. Let $\mathcal{A} \subset \mathcal{T}$ denote the full abelian subcategory generated by the representations $L(\lambda)$ for maximal atypical weights λ. In the following we

usually identify X^+ with the set of all labelings of the numberline, such that $\vee$ occurs r times for some $0 \leq r \leq n$ and $\times$ respectively $\circ$ occurs $m - r$ respectively $n - r$ times. There are two natural orderings on X^+, the Bruhat ordering and the coarser weight ordering. For $\lambda \in X^+$ let $\mathcal{T}^{\leq \lambda}$ respectively $\mathcal{T}^{<\lambda}$ denote the full subcategories of $\mathcal{T}$ generated by objects, all whose Jordan-Hölder constituents are simple modules $L(\mu)$ with highest weights $\mu \leq \lambda$ (resp. $\mu < \lambda$) with respect to the weight ordering.

The abelian category $\mathcal{T}$ decomposes into blocks Λ, defined by the eigenvalues of a certain elements in the center of the universal enveloping algebra of the Lie superalgebra $gl(m|n)$. Two irreducible representations $L(\lambda)$ and $L(\mu)$ are in the same block if and only if the weights λ and μ define labelings with the same position of the labels $\times$ and $\circ$. The degree of atypicality is a block invariant, and the blocks Λ of atypicality r are in 1-1 correspondence with pairs of disjoint subsets of $\mathbb{Z}$ of cardinality $m - r$ resp. $n - r$. The irreducible representations of each block Λ are in 1-1 correspondence to the subsets of cardinality r in the numberline with the subset of all $\times$ and all $\circ$ removed.

Example. The Berezin representation $Ber = Ber_{m|n}$ of $Gl(m|n)$ has highest weight $\lambda = (1, .., 1, 1.., 1; -1, ..., -1)$ with m digits 1 and n digits -1. Its superdimension is $sdim_k(Ber_{m|n}) = (-1)^n$ and its dimension is 1. All powers Ber^k for $k \in \mathbb{Z}$ of the Berezin are maximal atypical, and $L(\mu) \in \mathcal{A}$ iff $L(\mu) \otimes Ber^k \in \mathcal{A}$.

3   Maximal atypical blocks Λ

A block Λ is maximal atypical if and only if it does not contain any label $\circ$. Assume Λ is maximal atypical. Let j then be minimum of the subset of all x defined by Λ, or $j = 1$ if there is no cross. The uniquely defined weight λ, where the labels $\vee$ are at the positions $j - 1, ..., j - n$ is called the *ground state* of the block. There are higher ground states for $N = 0, 1, 2, 3, ..$, where the labels $\vee$ are at the positions $j - 1 - N, ..., j - N - n$. The corresponding weight vectors of these ground states are

$$\lambda_N = (\lambda_1, ..., \lambda_{m-n}, \lambda_{m-n} - N, ..., \lambda_{m-n} - N; -\lambda_{m-n} + N, ..., -\lambda_{m-n} + N)$$

where $\{\lambda_1, \lambda_2 - 1, ..., \lambda_{m-n} + 1 - m + n\}$ gives the positions of the labels $\times$. For $m = n$ these are the powers Ber^{-N} of the Berezin, and the ground state is the

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trivial representation 1. All ground states define irreducible representations $L(\lambda_N)$ in the given maximal atypical block Λ. For $\mu_i = \lambda_i - \lambda_{m-n}$, the representation

$$L(\lambda_0) \otimes Ber^{-\lambda_{m-n}} = L(\mu_1, ..., \mu_{m-n}, 0, ..., 0; 0, ..., 0) ,$$

again is an irreducible maximal atypical representation (usually in another block). It is covariant in the following sense:

Covariant representations. Let $\lambda_1 \geq \lambda_2 \geq ...$ be any partition λ of some natural number $N = deg(\lambda) := \sum_\nu \lambda_\nu$. Associated to this partion is the covariant representation

$$\{\lambda\} := Schur_\lambda(k^{m|n})$$

as a direct summand defined by a Schur projector of the N-fold tensor product $X^{\otimes N}$ of the standard representation X of $Gl(m|n)$ on $k^{m|n}$. The representation $\{\lambda\}$ so defined is zero iff $\lambda_{m+1} > n$, and is nontrivial and irreducible otherwise ([BR]). If the *hook condition* $\lambda_{m+1} \leq n$ is satisfied, we may visualize this by considering the Young diagram attached to λ with first column λ_1, second column λ_2 and so on. Let β denote the intersection of the Young diagram with the box $\{(x, y) | x \leq m, y \leq n\}$. Then λ has the following shape with subpartitions α, β, γ obtained by intersection we the three hook sectors

$$\lambda = \begin{matrix} & \alpha & \\ \beta & & \gamma \end{matrix} .$$

The transposed γ^* of the partition γ is again a partition with $(\gamma^*)_i = 0$ for $i > n$. We quote from [BR] and [JHKTM] the assertion of the next

Lemma 1. *If $\{\lambda\}$ is not zero, then $\{\lambda\} \cong L(\mu)$ is irreducible with highest weight μ defined by $\mu_i = \lambda_i$ for $i = 1, ..., m$, and $\mu_{m+i} = max(0, (\lambda^*)_i - m) = (\gamma^*)_i$ for $i = 1, ..., n$. In other words*

$$\left\{ \begin{matrix} \alpha & \\ \beta & \gamma \end{matrix} \right\} \cong L \begin{pmatrix} \alpha & \\ \beta & ; \gamma^* \end{pmatrix} .$$

This implies

Lemma 2. *A covariant representation $\{\lambda\}$ attached to a partion λ with the hook condition is maximal atypical if and only if $\lambda_{m-n+1} = 0$, and then $\mu = \lambda$ holds and*

$$\{\lambda\} \;=\; L(\lambda_1, ..., \lambda_{m-n}, 0, .., 0; 0, .., 0) \ .$$

Proof. One direction is clear. For $\lambda_{m-n+1} = 0$ the representation $\{\lambda\}$ corresponds to the ground state of some maximal atypical block as explained above. For the converse assertion notice that $I_\times(\mu) \subset [1-m, .., \infty)$ for the highest weight μ of the representation $\{\lambda\}$ by lemma 1. Hence $1 - m - \mu_{m+1}$ is in $I_\circ(\mu)$, but not in $I_\times(\mu)$ if $\mu_{m+1} = (\gamma^*)_1 > 0$. Hence, if $L(\mu)$ is maximal atypical, we conclude $\gamma^* = 0$ and hence $\gamma = 0$. So we can assume $m > n$. Then $I_\circ(\mu) = [1 - m, 2 - m, ..., n - m]$, since $\gamma^* = 0$. Therefore none of the $\mu_1, \mu_2 - 1, .., \mu_{m-n} + n - m + 1$ is in $I_\circ(\mu)$, since $\mu_i = \lambda_i \geq 0$ for $i = 1, .., m$. If $L(\mu)$ is maximal atypical, then all remaining $\mu_i + 1 - i$ for $i = m - n + 1, .., m$ must be contained in $I_\circ(\mu)$. Since $\lambda_{m-n+1} = \mu_{m-n+1}$ is ≥ 0, this implies $\lambda_{m-n+1} + n - m \in I_0(\mu)$ and therefore $\lambda_{m+n+1} = 0$. QED

Consider

$$\Pi \;=\; \Lambda^{m-n}(X) \otimes Ber_{m|n}^{-1} \;=\; L(0, .., 0, -1, .., -1; 1, .., 1)$$

with n digits 1 and -1, the first higher ground state in the $\mathbf{1}$-block.

Lemma 3. *Let Λ be a maximal atypical block. For the ground states $L(\lambda_N)$ of order $N = 0, 1, 2, ..$ in this block Λ we obtain*

$$L(\lambda_N) \otimes \Pi \;\cong\; L(\lambda_{N+1}) \oplus R_N$$

for certain R_N, whose projection to all maximal atypical blocks of $\mathcal{T}$ are zero.

Proof. By a twist with $Ber^{\lambda_{m-n} - N - 1}$ we can easily reduce to the case $N = 0$ and $\Lambda_{m-n+1} = 0$. Then $\lambda_N = \lambda$ for $\lambda = (\lambda_1, ..., \lambda_{m-n}, 0, .., 0; 0, .., 0)$ defines a covariant representation $L(\lambda_N) = \{\lambda\}$. By the well known properties of Schur projectors,

$$L(\lambda_N) \otimes \Lambda^{m-n}(X) \;=\; \bigoplus_\rho [\rho : \lambda, \Lambda^{m-n}] \cdot \{\rho\}$$

holds with the Littlewood-Richardson coefficients $[\rho : \lambda, \Lambda^{m-n}]$. It is well known that $[\rho : \lambda, \Lambda^{m-n}] \neq 0$ implies $\rho_{m-n+1} > 0$ unless $\rho = \lambda + \Lambda^{m-n}$. Hence by lemma 2 all summands $\{\rho\}$ are not maximal atypical, except for $\{\rho\} = \{\lambda + \Lambda^{m-n}\}$. By twisting with Ber^{-1} our claim follows. QED

4 The stable category $\mathcal{K}$

The abelian category $\mathcal{T} = \mathcal{T}_{m|n}$ is a Frobenius category, i.e. it has enough projective objects and the injective and projective objects coincide. Let $\mathcal{K} = \mathcal{K}_{m|n}$ be the quotient category with the same objects as $\mathcal{T}$, but with $Hom_{\mathcal{K}}(A, B)$ defined as the quotient of $Hom_{\mathcal{T}}(A, B)$ by the k-subvectorspace of all homomorphisms which factorize over a projective module. The natural functor

$$\alpha : \mathcal{T} \longrightarrow \mathcal{K}$$

is a k-linear tensor functor. The category $\mathcal{K}$ is a triangulated category with a suspension functor $S(A) = A[1]$, and the quotient functor α associates to exact sequences in $\mathcal{T}$ distinguished triangles in $\mathcal{K}$. Furthermore

$$Ext_{\mathcal{T}}^i(A, B) \cong Hom_{\mathcal{K}}(A, B[i]) \quad , \quad \forall\, i > 0 .$$

For simple X in $\mathcal{T}$ either X is projective in $\mathcal{T}$ and hence zero in $\mathcal{K}$, or X is not zero in $\mathcal{K}$ and

$$Hom_{\mathcal{K}}(X, X) = k \cdot id_X$$

is one dimensional. Notice that $A[1] \cong I/A$ for an embedding $A \hookrightarrow I$ into a projective object I, and similarly $A[-1] \cong Kern(P \to A)$ for a projective resolution $P \to A$. Since $P^* \cong P$ and $\mathcal{T}$ is a Frobenius category, this implies that * induces an involution of the stable category $\mathcal{K}$ such that $(A[n])^* \cong A^*[-n]$.

Theorem 1. *([BS2] corollary 5.15).* $dim_k(Ext_{\mathcal{T}}^i(L(\lambda), L(\mu)))$ *is equal to*

$$\sum_{j+k=i} \sum_{\nu} dim_k(Ext_{\mathcal{T}}^j(V(\nu), L(\lambda))) \cdot dim_k(Ext_{\mathcal{T}}^k(V(\nu), L(\mu))) .$$

Theorem 2. *([BS2], corollary 5.5).* $dim_k(Ext_{\mathcal{T}}^i(V(\lambda), L(\mu))) = 0$ *unless* $\lambda \leq \mu$ *and* $i \equiv l(\lambda, \mu)$ *modulo 2.*

Here $l(\lambda, \mu)$ denoted the minimum number of transpositions of neighbouring $\vee\wedge$ pairs needed to get from λ to μ, where neighbouring means separated only by $\circ$'s and x's. Put $p(\lambda) = \sum_{i=1}^n \lambda_{m+i}$. If λ and μ are maximal atypical, then $p(\lambda) \equiv p(\mu) + l(\mu, \nu)$ modulo two. Indeed it suffices to show this in the case of neighbours λ and μ where $l(\lambda, \mu) = 1$. For maximal atypical weights $I_x \cap I_0 = I_0$, a single transposition modifies $\sum_{j \in I_x \cap I_0} j = -\sum_{i=1}^n \lambda_{m+i} + \sum_{i=1}^n (i - m)$ by one. Hence the last two theorems imply

Lemma 4. *For $L(\lambda)$ and $L(\mu)$ in $\mathcal{A}$ and $i \geq 0$ we have $Hom_{\mathcal{K}}(L(\lambda), L(\mu)[i]) = 0$ unless $p(\lambda) \equiv p(\mu) + i$ modulo two.*

Lemma 5. *Let $\mathcal{A}$ be a k-linear category and let A and B be objects of $\mathcal{A}$ such that $End_{\mathcal{A}}(A) \cong k$ and $End_{\mathcal{A}}(B) \cong k$. Let $\varphi : B \to A$ and $\psi : A \to B$ be morphisms in $\mathcal{A}$. Then, either φ and ψ are isomorphisms, or $\varphi \circ \psi = 0$ and $\psi \circ \varphi = 0$.*

Proof. Suppose $\varphi \circ \psi \neq 0$. Then by a rescaling we can assume $\varphi \circ \psi = id_A$. Then $\varphi \circ \psi \circ \varphi = \varphi$. Since $\psi \circ \varphi = \lambda \cdot id_B$ for some $\lambda \in k$ and since $\varphi \neq 0$ by our assumption, we conclude $\lambda = 1$. Hence φ and ψ are isomorphisms. The same conclusions hold if $\psi \circ \varphi \neq 0$. QED

5 Kostant weights

By [BS2], lemma 7.2 a weight μ is a Kostant weight, i.e. satisfies for every $\nu \in \Lambda$

$$\sum_{i=0}^{\infty} dim_k(Ext^i_{\mathcal{T}}(V(\nu), L(\mu)) \ \leq \ 1 \ ,$$

if and only if no subsequence of type $\vee\wedge\vee\wedge$ occurs in its labeling. All ground states λ_N of the maximal atypical blocks are Kostant weights. In particular there is at most one index $i = i(\nu)$, depending on λ_N and ν, such that $Ext^i_{\mathcal{T}}(V(\nu), L(\lambda_N)) \neq 0$, and in this case $Ext^{i(\nu)}_{\mathcal{T}}(V(\nu), L(\lambda_N)) \leq 1$.

By [BS2] corollary 5.5 and the complementary formula (5.3) for $p_{\nu,\mu}$ in loc. cit. $Ext^i_{\mathcal{T}}(V(\nu), L(\mu)) = 0$ holds unless ν, μ are in the same block Λ, unless $\nu \leq \mu$ in the Bruhat ordering and $i \leq l(\nu, \mu)$. Suppose ν, μ are in the same block Λ, and suppose $\nu \leq \mu$ holds in the Bruhat ordering. Then for $i = l(\nu, \mu)$ we have $Ext^i_{\mathcal{T}}(V(\nu), L(\mu)) \neq 0$ by inspection of the formula (5.3) in [BS2]. Indeed in the set $D(\nu, \mu)$ of labeled cap diagrams C defined in loc. cit., there exists at least one cap diagram with $|C| = 0$, since $\nu \leq \mu$ in the Bruhat ordering implies $l_i(\nu, \mu) \geq 0$ for all $i \in I(\Lambda)$ in the notations of loc. cit. Since the leftmost vertex of a small cap is always $\vee$ and hence is contained in $I(\Lambda)$, there exists some $C \in D(\nu, \mu)$ with $|C| = 0$. This implies

Lemma 6. *For $L = L(\mu)$ and a Kostant weight μ the k-vectorspace $Ext^i_{\mathcal{T}}(V(\nu), L)$ is one dimensional, if ν, μ are in the same block Λ and $\nu \leq \mu$ holds in the Bruhat ordering and $i = l(\nu, \mu)$, and it is zero otherwise.*

If $\mu = \lambda_N$ is one of the ground state weights of the block Λ, then the conditions $\nu \in \Lambda$, $\nu \leq \mu$ and $i = i(\nu, \mu)$ only depend on the relative positions of the labels $\vee$ in the numberline after the crosses defined by the block Λ are removed. So they do not depend on the block, but only on the number n of labels $\vee$ of the Kostant weight. This implies

Corollary 1. *For any block Λ of $\mathcal{T}$ and ground state representation $L = L(\lambda_N)$ of this block we have for all $j \geq 0$*

$$dim_k(Ext_{\mathcal{T}}^j(L, L)) \;=\; dim_k(Ext_{\mathcal{T}}^j(\mathbf{1}, \mathbf{1})) \;.$$

Proof. By theorem 1 we get $dim_k(Ext_{\mathcal{T}}^j(L, L) = \sum_\nu dim_k(Ext_{\mathcal{T}}^{i(\nu)}(V(\nu), L))^2$ with summation over all ν such that $j = 2i(\nu)$. Since by lemma 6 the summation conditions and the dimensions in this sum only depend on j and not on the chosen ground state L or block Λ, we may replace L by the ground state $\mathbf{1}$ of the trivial block. QED

6   The localization $\mathcal{B}$ of the tensor category

For the moment let $\mathcal{K}$ be any k-linear triangulated tensor category (meaning symmetric monoidal). Then there exists a triangulated tensor functor $\mathcal{K} \to \mathcal{K}^\sharp$ of idemcompletion (see [BS]).

Let $k_{\mathcal{K}} = End_{\mathcal{K}}(\mathbf{1})$ be the central k-algebra of $\mathcal{K}$. Let $u \in \mathcal{K}$ be an invertible element (we use this for $u = \mathbf{1}[1]$). The monoidal symmetry $\sigma_u : u \otimes u \cong u \otimes u$ is given by multiplication with an element $\epsilon_u \in k_{\mathcal{K}}^*$ with $(\epsilon_u)^2 = 1$. Furthermore

$$R_{\mathcal{K}}^\bullet \;=\; \bigoplus_{i \in \mathbb{Z}} Hom_{\mathcal{K}}(\mathbf{1}, u^{\otimes i})$$

becomes a supercommutative ring with the parity ϵ_u, i.e. $fg = (\epsilon_u)^{ij} gf$ for homogenous elements of degree i and j. See [Ba], Prop. 3.3. Let $R \subset R_{\mathcal{K}}^\bullet$ be the graded subring generated by the elements of degree ≥ 0. Let $S \subset R_{\mathcal{K}}^\bullet$ be a multiplicative (and even, if $\epsilon_u \neq 1$) subset, then the ring localization $S^{-1}R_{\mathcal{K}}^\bullet$ is defined. Define a new category by the degree zero elements

$$Hom_{S^{-1}\mathcal{K}}(A, B) := (S^{-1}Hom_{\mathcal{K}}^\bullet(A, B))^0$$

of the localization of the graded $R_{\mathcal{K}}^{\bullet}$ module

$$Hom_{\mathcal{K}}^{\bullet}(A, B) \;=\; \bigoplus_{i \in \mathbb{Z}} Hom_{\mathcal{K}}(A, u^{\otimes i} \otimes B) \,.$$

For $M \in \mathcal{K}$ the annulator of $Hom_{\mathcal{K}}^{\bullet}(M, M))$ in R is a graded ideal, which defines the support variety $V(M)$ as the spectrum of the quotient ring.

Theorem 3. *([Ba], thm. 3.6). As a tensor category $S^{-1}\mathcal{K}$ is equivalent to the Verdier quotient of the tensor category $\mathcal{K}$ divided by the thick triangulated tensor ideal generated by the cones of the morphisms in S. The quotient category $\mathcal{B}$ is a $S^{-1}R_{\mathcal{K}}^{\bullet}$-linear category. The quotient functor is a k-linear triangulated tensor functor.*

In the following we only apply this for the stable category $\mathcal{K}$ of the representation category $\mathcal{T}$ for $u = \mathbf{1}[1]$. In this case $\epsilon_u = -1$. Then we have

Proposition 1. *([BKN1], 8.11). The graded ring R of the Lie super algebra $psl(n|n)$ is isomorphic to a graded polynomial ring $k[\zeta_2, \zeta_4, ..., \zeta_{2n-2}, \xi_n, \eta_n]$ of transcendency degree $n + 1$.*

By using restriction from $Gl(m|n)$ for $m > n$ one finds that for $Gl(n|n)$ the isomorphisms $\zeta_2, .., \zeta_{2n-2}$ also exist. Now suppose $m = n$. Then by proposition 1 there exists a power $\mathcal{L}$ of the Berezin such that $\xi_n : \mathbf{1} \to \mathcal{L}[n]$ and $\eta_n : \mathbf{1} \to \mathcal{L}^{-1}[n]$. The product $\zeta_{2n} = \eta_n \xi_n$ is in R. In the appendix 16 we show $\mathcal{L} = Ber_{n|n}$.

Proposition 2. *([BKN2], p.23). The graded ring R of the category $Gl(m|n)$ is isomorphic to a graded polynomial ring $S = k[\zeta_2, \zeta_4, ..., \zeta_{2n}]$ of transcendency degree n for all $m \geq n$.*

For the support variety $V(M)$ of an object $M \in \mathcal{K}$, being defined as above, we quote from [BKN1], section 7.2, p. 29 and [BKN2], 4.8.1

Theorem 4. *([BKN2], thm. 4.8.1). For $Gl(m|n)$ the dimension of the support variety of a simple object $L(\lambda)$ is equal to the degree of atypicality of $L(\lambda)$.*

Theorem 5. *([BKN2], thm. 4.5.1 and 4.8.1). For $Gl(m|n)$ the support variety of a simple maximal atypical object $L(\lambda)$ of $\mathcal{T}$ is $Spec(R)$.*

Now fix the supergroup $Gl(m|n)$. Put $K = Quot(R) = k(\zeta_2, ..., \zeta_{2n})$, and put $S = R\backslash\{0\}$. Notice, then as required S only contains even elements by proposition 2 above. Furthermore, since R is an integral domain, $S^{-1}R$ is isomorphic to the extension field K of k. Let $\mathcal{B} = R^{-1}\mathcal{K}$ be the corresponding localized category. If $\mathcal{B}$ is not idemcomplete, we replace it by its idemcompletion $\mathcal{B}^\sharp$ from now on. $\mathcal{B}$ is a K-linear category. It is obvious that the functor * respects R, hence induces a corresponding K-equivariant functor of $\mathcal{B}$. The natural quotient functor

$$\beta : \mathcal{K} \longrightarrow \mathcal{B}$$

is a k-linear triangulated tensor functor.

7 The homotopy category $\mathcal{H}$

In the last section we defined the quotient category $\mathcal{B}$ of the stable category $\mathcal{K}$. We now define another Verdier quotient category $\mathcal{H}$ of $\mathcal{K}$, which is obtained by dividing $\mathcal{K}$ by the thick triangulated $\otimes$-ideal of $\mathcal{K}$ generated by the anti Kac modules. Recall that for each highest weight $\lambda \in X^+$ there exists a cell module $V(\lambda)$ (or Kac module) in the sense of [BS4] in the category $\mathcal{T}$. We define the anti Kac modules by $V(\lambda)^*$ via the antiinvolution *.

Lemma 7. *For (anti)-Kac modules V and $\zeta_{2i} \in R$ the space $\bigoplus_{n\in\mathbb{Z}} Hom_{\mathcal{K}}(V, V[n])$ is annihilated by a sufficiently high powers of ζ_{2i}.*

Proof. By [BKN2], thm. 3.2.1 the groups $Ext^j_{\mathcal{T}}(V, M)$ vanish for fixed $M \in \mathcal{T}$ if $j >> 0$. Hence a high enough power of ζ_{2i} annihilates $Hom_{\mathcal{K}}(V, V[n])$. By applying the functor * this carries over to V^*. QED

Since ζ_{2i} becomes invertible in $\mathcal{B}$, this implies $Hom_{\mathcal{B}}(V, V) = 0$ for each Kac module V. Hence

Lemma 8. *The image of a (anti)-Kac module V is zero in $\mathcal{B}$.*

Recall that in [W] we defined the homotopy category $\mathcal{H}$, which is equivalent as a k-linear tensor category to the Verdier quotient of the category $\mathcal{K}$ devided by the thick tensor ideal generated by all anti-Kac modules. By the last lemma we obtain

Corollary 2. *The quotient functor $\beta : \mathcal{K} \to \mathcal{B}$ factorizes over the homotopy quotient functor $\gamma : \mathcal{K} \to \mathcal{H}$.*

We quote from [W] the following

Theorem 6. *In the category $\mathcal{H}$ for simple objects M and N in $\mathcal{T}$ with highest weights μ and λ the following holds:*

1. *$End_{\mathcal{H}}(M) = k \cdot id_M$,*

2. *$Hom_{\mathcal{H}}(M, N)$ is a finite k-vectorspace,*

3. *$Hom_{\mathcal{H}}(V, N) \cong Hom_{\mathcal{T}}(V, N)$ for every cell module $V = V(\mu)$.*

4. *$Hom_{\mathcal{H}}(M, N) = 0$ if $\mu < \lambda$ holds with respect to the weight ordering,*

5. *Let $\mathcal{H}^{\leq \lambda}$ denote the full subcategory quasi equivalent to the image of $\mathcal{T}^{\leq \lambda}$ and similar $\mathcal{H}^{<\lambda}$ for $\mathcal{T}^{<\lambda}$. Then suspension induces a functor*

$$[1] : \ \mathcal{H}^{\leq \lambda} \ \longrightarrow \ \mathcal{H}^{<\lambda} \ .$$

Lemma 9. $End_{\mathcal{B}}(1) = K$.

Proof. By [Ba], prop 3.3 we have $End_{\mathcal{B}}(1) = S^{-1}R_{\mathcal{K}}^{\bullet}$. Hence it suffices to show for $n > 0$ that all morphisms

$$\psi : \mathbf{1} \longrightarrow \mathbf{1}[-n]$$

are annihilated by a power of the element $\zeta_2 \in R$. Suppose $n = 2i - 1$ is odd. Then this is obvious, since $(\zeta_2)^i \circ \psi : \mathbf{1} \to u = \mathbf{1}[1]$ vanishes by parity reasons. Indeed $Hom_{\overline{\mathcal{T}}}(\mathbf{1}, \mathbf{1}[1]) = 0$ by lemma 4. Now suppose $n = 2i$ is even and consider $(\zeta_2)^i \cdot \psi$ in $End_{\mathcal{K}}(\mathbf{1}) = k$. By the trivial lemma 5 the morphism ψ is an isomorphism unless $(\zeta_2)^i \cdot \psi = 0$. If ψ were an isomorphism in $\mathcal{K}$, then also in $\mathcal{B}$ and therefore also in $\mathcal{H}$. However $Hom_{\mathcal{H}}(\mathbf{1}, \mathbf{1}[-n]) = 0$ holds for all $n > 0$ by the assertions 4) and 5) of theorem 6. QED

The same argument shows

Lemma 10. *For simple objects M in $\mathcal{T}$ the image of $Hom_{\mathcal{K}}(M, M[-n])$ under the natural map*

$$Hom_{\mathcal{K}}(M, M[-n]) \longrightarrow S^{-1}Hom_{\mathcal{K}}^{\bullet}(M, M)^0 = Hom_{\mathcal{B}}(M, M)$$

is zero for $n > 0$.

Proof. $Hom_{\mathcal{H}}(M, M[-n])$ vanishes for simple M and all $n > 0$ by weight reasons. Use part 4) and 5) of theorem 6. QED

By lemma 10 for simple objects M the endomorphism ring $Hom_{\mathcal{B}}(M, M)$ is obtained by quotients f/r for $f : M \rightarrow M[i]$ for $f \in Hom_{\mathcal{K}}(M, M[i])$ and $r : \mathbf{1} \rightarrow \mathbf{1}[i]$ in $r \in Hom_{\mathcal{K}}(\mathbf{1}, \mathbf{1}[i])$ only for positive degrees $i \geq 0$, of course modulo the usual equivalence defined by the localization S^{-1}. We may suppose that the simple object M is not projective, since otherwise M vanishes in $\mathcal{K}$ and hence in $\mathcal{B}$. Then $M \neq 0$ in $\mathcal{M}$, and $Hom_{\mathcal{T}}(M, M) \cong Hom_{\mathcal{K}}(M, M) \cong k$. Hence $Ann_R(Ext_{\mathcal{T}}^{\bullet}(M, M)) = Ann_R(\bigoplus_{i=0}^{\infty} Hom_{\mathcal{K}}(M, M[i]))$. Hence $r \in Ann_R(Ext_{\mathcal{T}}^{\bullet}(M, M))$ iff $r \cdot f = 0$ for all $f \in Ext_{\mathcal{T}}^{\bullet}(M, M)$. This is related to the support variety $V(M)$ of the simple object M

$$V(M) = Spec(R/Ann_R(Ext_{\mathcal{T}}^{\bullet}(M, M)))$$

as follows. There exists an $r \neq 0$ in R that annihilates $\bigoplus_{i=0}^{\infty} Hom_{\mathcal{K}}(M, M[i])$ iff the support of $V(M)$ is not equal to $Spec(R)$. The first statement is equivalent to $Hom_{\mathcal{B}}(M, M) = 0$. Hence

Corollary 3. *Let M be a simple object of $\mathcal{T}$. Then M vanishes in $\mathcal{B}$ if and only if the support variety $V(M)$ is a proper subset of $Spec(R)$.*

Corollary 4. *A simple object M of $\mathcal{T}$ vanishes in $\mathcal{B}$ iff M is not maximal atypical.*

Proof. Use theorem 4 and 5.

Lemma 11. *For simple objects M and N in $\mathcal{T}$ the space $Hom_{\mathcal{B}}(M, N)$ vanishes unless M and N have the same parity in the sense of lemma 4.*

Proof. As explained above, any morphism in $\mathcal{B}$ between M and N is of the form f/r for $f : M \to N[i]$ in $\mathcal{K}$, $r \in R$ for some even $i \geq 0$. Hence f corresponds to an element in $Ext^i_{\mathcal{T}}(M, N)$. We can assume that M and N are maximal atypical, since otherwise M and N are zero in $\mathcal{B}$. Then $Ext^i_{\mathcal{T}}(M, N)$ vanishes by lemma 4 unless M and N have the same parity. QED

We remark that ζ_2 becomes an isomorphism in $\mathcal{B}$. Hence

Lemma 12. $\mathbf{1}[2] \cong \mathbf{1}$ *in* $\mathcal{B}$.

8   The ground state categories $\mathcal{Z}_\Lambda$

Definition. Let Λ be a maximal atypical block of $\mathcal{T}$ and L be the ground state representation of this block. Let $\mathcal{Z}_\Lambda$ denote the full subcategory of $\mathcal{B}$ of all objects isomorphic to a finite direct sum of L and $L[1]$. For the unit block where $L = \mathbf{1}$ we simply write $\mathcal{Z}$.

Lemma 13. $\mathcal{Z}_\Lambda$ *is a thick idemcomplete triangulated subcategory of $\mathcal{B}$, i.e. it is closed under retracts and extensions and the shift functor.*

Proof. First suppose $L = \mathbf{1}$. Notice

$$Hom_\mathcal{B}(a \cdot \mathbf{1} \oplus b \cdot \mathbf{1}[1], c \cdot \mathbf{1} \oplus d \cdot \mathbf{1}[1]) \cong Hom_K(K^a, K^c) \oplus Hom_K(K^b, K^d)$$

by lemma 4 and lemma 9. Hence by usual properties of matrix rings the category $\mathcal{Z}$ is an idempotent split category. By the same reasons $\mathcal{Z}$ is closed under retracts as well as under cones. Since $\mathbf{1}[2] \cong \mathbf{1}$, shifts preserve $\mathcal{Z}$. Hence $\mathcal{Z}$ is a thick idempotent split triangulated subcategory of $\mathcal{B}$. The same carries over to $\mathcal{Z}_\Lambda$ by the next lemma. QED

Lemma 14. *Suppose L is a ground state of a maximal atypical block Λ in $\mathcal{T}$. Then $Ext^\bullet_{\mathcal{T}}(L, L)$ is a graded free module over $R = Ext^\bullet_{\mathcal{T}}(\mathbf{1}, \mathbf{1})$, hence in particular*

$$End_\mathcal{B}(L) \cong K \cdot id_L \,.$$

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Proof. By theorem 5 the annihilator of $Ext^\bullet_\mathcal{T}(L, L)$ in R is trivial. Hence the graded R-homomorphism $R \cdot id_L \to Ext^\bullet_\mathcal{T}(L, L)$ is injective. By corollary 1 the comparison of k-dimensions shows that it is an isomorphism. QED

Lemma 15. *Suppose the image of an irreducible maximal atypical highest weight representation $L(\lambda)$ in $\mathcal{B}$ is contained in $\mathcal{Z}$. Then*

$$sdim_k(L(\lambda)) = (-1)^{p(\lambda)} \cdot m(\lambda)$$

for some integer $m(\lambda) > 0$.

Proof. By corollary 4 the object $L(\lambda)$ is not zero in $\mathcal{B}$. Lemma 4 together with the assumption $L(\lambda) \in \mathcal{Z}$ hence implies $L(\lambda) \cong c \cdot \mathbf{1}[p'(\lambda)]$ for some $c \neq 0$ in $\mathbb{Z}$, where $p'(\lambda) \in \{0, 1\}$ is uniquely defined. Then $sdim_k(L(\lambda)) = (-1)^{p'(\lambda)}c$. Two remarks. First $\chi_\mathcal{H}(\mathbf{1}[1]) = \chi_\mathcal{B}(\mathbf{1}[1]) = -1$ for $u = \mathbf{1}[1]$, since $\epsilon_u = -1$. Notice, this holds in the homotopy category $\mathcal{H}$ by [W] and hence in $\mathcal{B}$. Secondly for all $L(\lambda)$ in $\mathcal{Z}$ we have $p'(\lambda) = p(\lambda) + const$, where $const \in Z$ is independent from $L(\lambda)$ in $\mathcal{Z}$ by lemma 4. For $\lambda = 0$ and $L(\lambda) = 1$ this shows $const \in 2\mathbb{Z}$. Therefore $(-1)^{p'(\lambda)} = (-1)^{p(\lambda)}$.

Lemma 3 and corollary 4 imply for the atypical representations $L(\lambda_N), L(\lambda_{N+1})$ and Π the following relation in $\mathcal{B}$

Lemma 16. *In $\mathcal{B}$ we have $L(\lambda_N) \otimes \Pi \cong L(\lambda_{N+1})$ for all $N \geq 0$.*

9 The reductive group $Gl(m - n)$

Consider the tensor category $\mathcal{T} = \mathcal{T}_{m|n}$ as before. Let $H = Gl(m - n)$ denote the linear group over k and let $Rep_k(H)$ denote the k-linear rigid semisimple tensor category of all algebraic representations of $Gl(m - n)$ on finite dimensional k-vectorspaces.

Embedded in $Gl(m|n)$, with an immersion in the obvious way, is subgroup $H \times Gl(n|n)$. Restriction of a super representation of $Gl(m|n)$ on a finite dimensional k-super vectorspace to the subgroup $H \times Gl(n|n)$ defines a k-linear exact tensor functor

$$Res : \mathcal{T}_{m|n} \longrightarrow Rep_k(H) \otimes_k \mathcal{T}_{n|n} \ .$$

The restriction of a projective representation P in $\mathcal{T}_{m|n}$ decomposes into a direct sum of isotypic representations $P = \bigoplus_\rho P_\rho$ with respect to the action of the reductive group H. Each P_ρ is a $Gl(n|n)$ representation, which is projective as a direct summand of the projective object P viewed as a super representation in $\mathcal{T}_{n|n}$. Hence $Res(\mathcal{P}) \subset Rep_k(H) \times \mathcal{P}$. Similarly a morphisms in $\mathcal{T}_{m|n}$, which is stably equivalent to zero, restricts to a direct sum of morphisms f_ρ with respect to the action of H, such that each of the morphisms f_ρ is stably equivalent to zero in $\mathcal{T}_{n|n}$. Hence the restriction induces a tensor functor

$$res : \ \mathcal{K}_{m|n} \ \longrightarrow \ Rep_k(Gl(m-n)) \otimes_k \mathcal{K}_{n|n} \ .$$

The suspension $(.)[1]$ thereby maps to the suspension $id_{Rep_k(H)} \otimes_k (.)[1]$, since an embedding $X \hookrightarrow I$ decomposes into $Res(X) = \bigoplus_\rho Res(X)_\rho \hookrightarrow \bigoplus_\rho Res(I)_\rho$. Thus res becomes a triangulated k-linear tensor functor. The triangulated structure on $Rep_k(Gl(m-n)) \otimes_k \mathcal{K}_{n|n}$ is induced by the triangulated structure on $\mathcal{K}_{n|n}$ in an obvious way, noticing that $Rep_k(H)$ is semisimple.

Using detecting subalgebras as in [BKN] one can show that $Ext^n_{\mathcal{T}_{m|n}}(k,k)$ restricts properly and surjectively to $Ext^n_{\mathcal{T}_{n|n}}(k,k)$. By the universal property of the Verdier quotient categories the functor res induces a functor from the Verdier quotient categories $\mathcal{B}_{m|n}$ of $\mathcal{T}_{m|n}$

$$\gamma : \mathcal{B}_{m|n} \ \longrightarrow \ Rep_k(H) \otimes_k \mathcal{B}_{n|n} \ .$$

This functor is a K-linear triangulated tensor functor.

Now use

Theorem 7. *For $m \geq n$ the image of the block of the trivial representation in the category $\mathcal{B}_{m|n}$ is equivalent as a K-linear triangulated tensor subcategory of $\mathcal{B}_{m|n}$ to the K-linear triangulated tensor category $\mathcal{Z} \sim svec_K$ of finite dimensional K-super vectorspaces.*

Taking this theorem for granted at the moment we proceed as follows: We apply the last theorem for $m = n$, which allows us to consider γ as a functor K-linear triangulated tensor functor

$$\gamma : \mathcal{B}_{m|n} \ \longrightarrow \ Rep_k(H) \otimes_k svec_K \ .$$

The right side can be viewed as the category of finite dimensional K-algebraic super representations of the reductive group $Gl(m-n)$ over K. Up to twist by powers of the determinant representation of $Gl(m-n)$ a basis of simple objects is given by the representations $Schur_\mu(K^{m-n})$ and $Schur_\mu(K^{m-n})[1]$, where $\mu = \mu_1 \geq \mu_2 \geq ... \geq \mu_{m-n} \geq 0$ runs over the partitions of length $\leq m-n$.

Let us look what the functor γ does with the image in $\mathcal{B}_{m|n}$ of the standard representation $X_{m|n} = k^{m|n}$ of $Gl(m|n)$. This standard representation restricts to $Res(X_{m|n}) = (k^{m-n} \otimes_k \mathbf{1}) \bigoplus (\mathbf{1} \otimes_k X_{n|n})$. $X_{n|n}$ becomes zero in $\mathcal{B}_{n|n}$ by lemma 4, since it is non maximal atypical. Hence

$$\gamma(X_{m|n}) \cong K^{m-n}$$

is the standard K-linear representation of $Gl(m-n,K)$ on K^{m-n}. Now we use the following stronger version of the last theorem

Theorem 8. *As a K-linear triangulated category the full image of each block Λ of $\mathcal{T}_{m|n}$ in $\mathcal{B}_{m|n}$ is isomorphic to the K-linear triangulated category $svec_K$ spanned by the ground state $L(\lambda_0)$ of the block Λ.*

Theorem 8 immediately implies theorem 7. Since γ is a K-linear triangulated tensor functor, theorem 8 also implies that γ is an exact K-linear equivalence of K-linear triangulated abelian tensor categories once we know

Lemma 17. *The category $\mathcal{B}_{m|n}$ is semisimple and hence abelian.*

Indeed the lemma implies exactness of the functor γ, and then corollary 4 and theorem 8 imply faithfulness. Hence φ induces a faithful embedding of categories. That γ is full then is an immediate consequence. Put $\varphi = \gamma \circ \beta \circ \alpha$. Then theorem 8 and lemma 17 imply the following generalization of theorem 7

Main Theorem. *As a rigid K-linear triangulated tensor category $\mathcal{B}_{m|n}$ is semisimple and hence abelian, and as a K-linear abelian tensor category $\mathcal{B}_{m|n}$ is equivalent to the category of K-algebraic finite dimensional K-linear super representations of the reductive K-group $Gl(m-n)$. Each simple maximal atypical object $M = L(\lambda)$ maps to*

$$\varphi(M) = m(\lambda) \cdot L(\lambda_0)\,[p(\lambda)]\,,$$

where the multiplicity $m(\lambda)$ is an integer > 0 and $L(\lambda_0)$ denotes the ground state in the block Λ defined by $L(\lambda)$.

Remark. In particular this confirms the conjecture of Kac and Wakimoto in the case of superlinear groups.

Remark. For the object $\Pi_{m|n} = \Lambda^{m-n}(X) \otimes Ber^{-1}$ we have

$$\varphi(\Pi_{m|n}) \;=\; \mathbf{1}_{m-n} \otimes (Ber_{n|n})^{-1}$$

in $Rep_k(H) \otimes_k \mathcal{B}_{n|n}$. By the main theorem $\mathcal{B}_{m|n} \cong Rep_k(H) \otimes_k \mathcal{B}_{n|n}$. Hence $\Pi_{m|n}$ is invertible in the tensor category $\mathcal{B}_{m|n}$. Since $Ber_{n|n}$ is invertible in $\mathcal{B}_{n|n} \cong svec_K$ it follows that $Ber_{n|n} \cong \mathbf{1}[p(Ber_{n|n})] = \mathbf{1}[n]$, hence $Ber_{n|n} \cong \mathbf{1}[n]$ in $\mathcal{B}_{n|n}$. Therefore theorem 8 and lemma 17 imply

Corollary 5. *The object* $\Pi = \Pi_{m|n} = \Lambda^{m-n}(X) \otimes (Ber_{m|n})^{-1}$, *where* $X = k^{m|n}$ *is the standard representation, becomes isomorphic to* $\mathbf{1}[n]$ *in the triangulated tensor category* $\mathcal{B}_{m|n}$

$$\Pi_{m|n} \;\cong\; \mathbf{1}[n] \ .$$

By lemma 16 this in turn implies

Corollary 6. $L(\lambda_N) \;\cong\; L(\lambda_{N+1})[n]$ *in* $\mathcal{B}$.

Proof of lemma 17 using theorem 8. This lemma follows from corollary 9 of the section 17, since the conditions for this corollary are provided by the parity lemma 4 and theorem 8, which will be proved in the next sections 10, 11 and 12.

10 Basic moves

We consider blocks Λ for the group $Gl(m|n)$ of maximal atypical type. As explained in section 2 they are described by an associated set of $m-n$ crosses $\times$ on the numberline $\mathbb{Z}$. The weight λ in this block is uniquely described by n labels $\vee$, which are at position different from the crosses. Attached to a weight λ is its cup diagram $\underline{\lambda}$ (right move) and the oriented cup diagram $\underline{\lambda}\lambda$.

Some simplification. In the cup diagrams of [BS1] for many arguments the crosses $\times$ often do not play a role. This is also true for our discussion below. Hence, for

the simplicity of exposition, we often assume $m = n$ in this section, although all statements hold for $m \geq n$ without changes. So assume $m = n$. Then $\mathcal{B} = \mathcal{B}_{n|n}$, so that there are no crosses for maximal atypical weights. The n labels $\vee$ attached to a maximal atypical weight define a subset $J = \{x_1, .., x_n\}$ of the numberline $\mathbb{Z}$. We order the integers such that $x_1 > ... > x_n$ and put $\lambda_j = x_j + j - 1$. Then $\lambda = (\lambda_1, .., \lambda_n; -\lambda_n, .., -\lambda_1)$ gives the associated weight vector of a maximal atypical simple object $L(\lambda)$.

Sectors and segments. Every cup diagram for a weight with n labels $\vee$ contains n lower cups. Some of them may be nested. If we remove all inner parts of the nested cups there remains a cup diagram defined by the (remaining) outer cups. We enumerate these cups from left to right. The starting points of the j-th lower cups is denoted a_j, its endpoint is denoted b_j. Then there is a label $\vee$ at the position a_j and a label $\wedge$ at position b_j. The interval $[a_j, b_j]$ of the numberline will be called the j-th sector of the cup diagram. Adjacent sectors, i.e with $b_j = a_{j+1} - 1$ will be grouped together into segments. The segments again define intervals in the numberline. Let s_j be the starting point of the j-th segment and t_j the endpoint of the j-th segment. Between any two segments there is a distance at least ≥ 1. The interior I^0 of a sector, which is obtained by removing the start and end point of the sector, always is a segment. Hence sectors, and therefore also segments have even length.

Example $n = 2$. For the weight

$$\mu \quad ... \ \wedge\wedge\wedge\wedge\vee\vee\wedge\wedge\wedge\wedge \ ... \ ,$$

with labels $\vee$ at the positions $j, j+1$ and all other labels equal to $\wedge$, the cup diagram $\underline{\mu}$ is described by one segment (which is a single sector)

$$[j, j+1, j+2, j+3] .$$

Graphically it corresponds to a nested pair of outer cups, one from $j + 1$ to $j + 2$, and one below from j to $j + 3$.

Now we fix some weight, which we denote $\lambda_{\vee\wedge} = (\lambda_1, .., \lambda_n, -\lambda_n, .., -\lambda_1)$ for reasons to become clear immediately. For the weight $\lambda_{\vee\wedge}$ we pick one of the labels $x_j \in J$ at the position $i := x_j$ such that $i + 1$ is not contained in the set of labels J of the weight $\lambda_{\vee\wedge}$. Equivalently this means $\lambda_j < \lambda_{j+1}$ in terms of the weight vector. We define a new weight λ (which is in another block, and in

particular is not maximal atypical) by replacing in $\lambda_{\vee\wedge}$ the label $\vee$ at the position i by a cross x, and the label $\wedge$ at the position $i+1$ by a circle $\circ$. Attached to this new weight λ is an irreducible, but not maximal atypical representation $L(\lambda)$.

Now consider the functor F_i defined in [BS4] on p.6ff and p.10ff, which is attached to the admissible matching diagram t
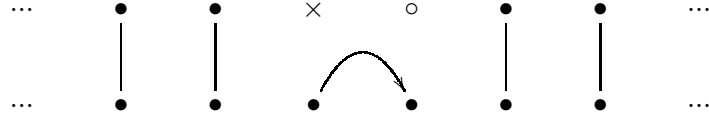
with $\times$ at position i and $\circ$ at position $i+1$, and the maximal atypical object

$$F_i(L(\lambda)) \;=\; \mathbf{F}_\lambda \;.$$

According to [BS2], lemma 4.11 this object is indecomposable and maximal atypical with irreducible socle and cosocle isomorphic to $L(\lambda_{\vee\wedge})$.

Lemma 18. *The Loewy diagram of $\mathbf{S}_\lambda$ looks like*

$$
\begin{array}{c}
L(\lambda_{\vee\wedge}) \\
| \\
F_\lambda \\
| \\
L(\lambda_{\vee\wedge})
\end{array}
$$

with a semisimple object F_λ in the middle.

For the proof we give a description of the simple constituents of F_λ below using [BS4] case (v), subcase (b), which shows that all of these constituents have the same parity (different from the parity of $\lambda_{\vee\wedge}$). This suffices to show the claim that F_λ is semisimple, using lemma 4. QED

Next we quote from [BS4] formula (2.13) and corollary 2.9 (of course for arbitrary $m \geq n$)

Lemma 19. $\mathbf{F}_\lambda$ *is a direct summand of the representation $L(\lambda) \otimes X$, where X denotes the standard representation on $k^{m|n}$.*

Since $L(\lambda)$ is not maximal atypical, it becomes trivial in $\mathcal{B}$ by corollary 4. Hence the same holds for the tensor product $L(\lambda) \otimes X$, and any of its direct summands.

Corollary 7. $\mathbf{F}_\lambda \cong 0$ *in $\mathcal{B}$.*

Corollary 8. $F_\lambda[1] \cong L(\lambda_{\vee\wedge}) \oplus L(\lambda_{\vee\wedge}) = 2 \cdot L(\lambda_{\vee\wedge})$.

Proof. Corollary 7 gives a distinguished triangle in $\mathcal{B}$

$$L(\lambda_{\vee\wedge})[-1] \to F_\lambda \to L(\lambda_{\vee\wedge})[1] \to L(\lambda_{\vee\wedge})$$

whose last arrow vanishes by lemma 4. This proves the claim, since $1[-1] \cong 1[1]$ holds in $\mathcal{B}$.

The rules of [BS2], theorem 4.11. The constituents of $\mathbf{F}_\lambda$ correspond to the maximal atypical weights μ with defect n such that

1. The (unoriented) cup diagram $\underline{\lambda}$ is a lower reduction of the oriented cup diagram $\underline{\mu}t$ for our specified matching diagram t.

2. The rays in each "lower line" in the oriented diagram $\underline{\mu}\mu t$ are oriented so that exactly one arrow is $\vee$ and one arrow is $\wedge$ in each such line.

3. μ appears with the multiplicity $2^{n(\mu)}$ as a constituent of $\mathbf{F}_\lambda$, where $n(\mu)$ is the number of "lower circles" in μt.

We remark that the lower reduction (for more details see [BS] II, p.5ff) is obtained by removing all "lower lines" and all "lower circles" of the diagram μt, i.e. those which do not cross the upper horizontal numberline.

Let I be the set of labels $\vee$ defining the maximal atypical weight $\lambda_{\vee\wedge}$. Then $i \in I$. To evaluate these conditions in more detail consider the segment J of I containing $i \in I$. Then also $i + 1 \in J$. Notice that J is an interval. This segment decomposes into a disjoint union of sectors, which completely cover the interval J. We distinguish two cases

The unencapsulated case. Here the interval $[i, i + 1]$ is one of the sectors of J. We write $J = [a + 1, ..., i, i + 1, ..., b - 1]$ for the segment and call a and b the left and right boundary lines of the segment. Then the label of λ at a and b must be $\wedge$ by definition. We write $I = [a, .., b]$.

The encapsulated case. By definition this means that the interval $[i, i + 1]$ lies nested inside one of the sectors of J. Hence there exists a maximal $a < i$ defining a left starting point of a cup within the cup diagram of λ, that has right end point b such that $i + 1 < b$. We write $I = [a, ..., i, i + 1, ..., b]$ for this subinterval of J and call a and b the left and right boundary of I. The label of λ at a is $\vee$ and the label at b is $\wedge$ by definition.

In both cases consider the sectors within $I^0 = [a + 1, ..., b - 1]$. By the maximality a a $[i, i + 1]$ is one of the sectors of I^0. The remaining sectors to the left of $[i, i + 1]$ and to the right of $[i, i + 1]$ will be called the lower and upper internal sectors. Let a_j denote the left starting points and b_j the right ending point of the j-th internal sector. The labels at the points a_j are $\vee$, and the labels at the points b_j are $\wedge$. There may be no such internal upper or lower sectors. If there are, then we will see that to each of them corresponds an irreducible summand $L(\mu)$ of S_λ, which we will see is uniquely described by the corresponding internal sector.

We summarize. In both cases the interval I^0 is completely filled out by the disjoint union of the internal sectors, and one of these internal sectors is $[i, i + 1]$.

List of summands of $\mathbf{F}_\lambda$.

- *Socle and cosocle.* They are defined by $L(\mu)$ for $\mu = \lambda_{\vee\wedge}$.

- *The upward move.* It corresponds to the weight $\mu = \lambda_{\wedge\vee}$ which is obtain from $\lambda_{\vee\wedge}$ by switching $\vee$ and $\wedge$ at the places i and $i + 1$. It is of type $\lambda_{\wedge\vee}$.

- *The nonencapsulated boundary move.* It only occurs in the nonencapsulated case. It moves the $\vee$ in $\lambda_{\vee\wedge}$ from position i to the left boundary position a. The resulting weight μ is of type $\lambda_{\wedge\wedge}$.

- *The internal upper sector moves.* For every internal upper sector $[a_j, b_j]$ (i.e. to the right of $[i, i + 1]$) there is a summand whose weight is obtained from $\lambda_{\vee\wedge}$ by moving the label $\vee$ at a_j to the position $i + 1$. These moves define new weights μ of type $\lambda_{\vee\vee}$.

- *The internal lower sector moves*. For every internal lower sector $[a_j, b_j]$ (i.e. to the left of $[i, i+1]$) there is a summand whose weight is obtained from $\lambda_{\vee\wedge}$ by moving the label $\vee$ from the position i to the position b_j. These moves define new weights μ of type $\lambda_{\wedge\wedge}$.

Proof of lemma 18. Except for the first in the list of summands of $\mathbf{F}_\lambda$, the moves of this list define the weights μ of the constituents $L(\mu)$ of F_λ. The parity of these weights μ is always different from $\lambda_{\vee\wedge}$. The reason for this is, that sectors always have even length. The unique label $\vee$ changing its position during the move, is moved by an odd number of steps. As already explained, this suffices to prove lemma 18. QED

Remark 1. All upper and lower internal sector moves change the weight $\lambda_{\vee\wedge}$ into weights μ, whose cup diagram restricted to I^0 has a strictly smaller number of sectors. Hence in the nonencapsulated case, the full cup diagram of any of these μ has a strictly smaller number of sectors than the cup diagram of $\lambda_{\vee\wedge}$.

Remark 2. Similarly, the nonencapsulated boundary move changes the starting weight $\lambda_{\vee\wedge}$ into a weight μ, whose cup diagram has a strictly smaller number of sectors except for the case $a = i - 1$ (where $a = i - 1$ is equivalent to $\lambda_{j-1} < \lambda_j$).

Remark 3. Except for the first case in the list of summands of $\mathbf{F}_\lambda$, all other moves belong to diagrams without "lower circles". Hence $n(\mu) = 1$ holds in these cases.

Remark 4. In the encapsulated case the diagrams $\underline{\mu}t$ do not contain "lower lines".

11 Three algorithms

For $Gl(m|n)$ we discuss now three algorithms, which can be successively applied to a cup diagram of some maximal atypical weight within a block Λ to reduce this weight to a collection of the ground state weights of this block Λ which have the form $(\lambda_1, .., \lambda_{m-n}, -N, ..., -N; N, ..., N)$ for certain large integers $N \geq 0$. Notice, the integers $\lambda_1, .., \lambda_{m-n}$ are fixed and describe the given block Λ.

In fact, since these algorithms applies within a fixed maximal atypical block Λ, it suffices to describe these algorithms in the case $m = n$. This simplifies the exposition. For this purpose assume $m = n$.

Algorithm I. The first algorithm deals with a union of different segments. The aim is to move all labels $\vee$ to the left in order to eventually reduce everything to a single segment. For a given maximal atypical weight λ let $S_j = [s_j, t_j]$ from left to right denote the segments of its cup diagram $\underline{\lambda}$. Let denote $0 \leq c_j = \#S_j \leq n$ their cardinalies and let denote $-\infty \leq d_j = 1 - |s_{j+1} - t_i| \leq 0$ the negative distance between two neighbouring segments. We endow the set C of pairs of integers $\gamma = (c, d)$ with the lexicographic ordering. Next we endow the set C^n of all $((c_1, d_1), (c_2, d_2), ...) = (\gamma_1, \gamma_2, ...)$ with the corresponding induced lexico-graphical ordering. A cup diagram defines a maximal element in this ordering if and only if it contains a single segment, in which case $\gamma_1 = (n, -\infty)$.

Claim. Moving the starting point of the second segment to the left increases the ordering. To be more precise: Suppose there exist at least two segments in the cup diagram. Put $i = s_2 - 1$ and $i + 1 = s_2$ and the weight $\lambda_{\vee\wedge}$ obtained from the given weight $\lambda_{\wedge\vee}$ defining the cup diagram c by interchange at i and $i + 1$. Then $[i, i+1]$ is a sector of the new cupdiagram c' obtained in this way attached to $\lambda_{\vee\wedge}$. Let $[a_j, b_j]$ denote the sectors of the second segment S_2 with $a_1 = s_1$. There are two possibilities:

- Then $[i, i+1][a_1 + 1, b_1 - 1]$ is a segment of c' (namely the second segment, whose first sector is $[i, i+1]$. This is the case if and only if $d_1 < -1$;

- or $[s_1, .., t_1][i, i+1][s_2 + 1, .., b_1 - 1]$ is the first segment of c'. This is the case if and only if $d_1 = -1$.

In the first case $s_1' = s_1$ but $d_1' > d_1$. In the second case $s_1' = s_1 + 1$. Hence c' is larger than c with respect to our ordering.

Now we consider the (unencapsulated) move centered at $[i, i+1]$ for the cup diagram c_0' attached to the weight $\lambda_{\vee\wedge}$. Moving up gives the cup diagram c we started from. The down move, corresponding to the left boundary move, either gives as second segment $[i - 1, i]$ with unchanged first segment. Or, if $d_1 = -1$, it increases the cardinality of the first segment to $s_1 + 1$. The same holds for all internal lower sector moves. Finally for the internal upper sector moves. All these moves give cup diagrams of the following type: With second segment $[i, i+1][a_2 + 1, ..,]..[..b_2 - 1]$ if $d_1 < -1$ or with first segment $[s_1, .., t_1][i, i+1][a_2 + 1, ..,]..[..b_2 - 1]$ if $d_1 < -1$. Indeed they all have the same segment structure as c_0', but different sector structure. However we see that algorithm I relates the given cup diagram c to a finite number of cup diagrams c' such that $c' > c$ in our lexicographic ordering.

Algorithm II. Decreasing the number of sectors within a segment. Suppose c is a maximal atypical cup diagram attaches to a weight $\lambda_{\wedge\vee}$ with only one segment. Let $[a_j, b_j]$ for $j = 1, .., r$ denote its sectors, counted from left to right. Assume there are at least two sectors, i.e. assume $r > 1$. Put $i = b_j$ and $i + 1 = a_{j+1}$ for some $1 \leq j < r$. For this recall, that any sector starts with a $\vee$ and ends with a $\wedge$. Define $\lambda_{\vee\wedge}$ by exchanging the position of $\vee$ and $\wedge$ in $\lambda_{\wedge\vee}$ at i and $i + 1$. This gives a new cup diagram c'_0. It has only one segment, the same as the segment of c. However the j-th and the $j + 1$-th sectors have become melted in c'_0 into one single sector. The other sectors remain unchanged. So the numbers of sectors in the segment decreases by one. Now consider the (encapsulated) move at $[i, i + 1]$ starting from the cup diagram $c'0$. Its move up gives the cup diagram c we started from. All internal lower and upper moves occur within the sector $[a_j, .., b_{j+1}]$, i.e. the lower bound $a \geq a_j$ and the upper bound is $b \leq b_{j+1}$. None of these moves changes the cup starting from a_j and ending in b_{j+1}. Hence the internal moves all yield cup diagrams with the same sector structure as c'. Hence algorithm II relates the given cup diagram c (with one segment S and r sectors) to a finite number of cup diagrams c', each of them with the same segment S but with $r - 1$ sectors.

Algorithm III. Now assume c is a cup diagram with one segment, which consists of a single sector $[a, ..., b]$. The sector cup from a to b encloses an internal cup diagram with $n - 1$ labels $\vee$. This internal cup diagram necessarily defines one segment, namely the segment $[a + 1, .., b - 1]$. We now apply algorithm II to this internal segment. This finally ends up into some Kostant weights (see [BS] II, lemma 7.2 and section 5)

Further iteration. We remark that we can start all over again and move the left starting point of the sector of a Kostant weight further to the left using algorithm I, and then repeat the whole procedure of applying algorithms I, II and III. At the end this allows to replace the given Kostant weight by some other Kostant weights further shifted to the left on the numberline (with all crosses $\times$ removed in case $m \geq n$). If we repeat this down shift of Kostant weights sufficiently often we end up with a bunch of Kostant weights, that are ground states of the block, i.e. whose associated irreducible representation is one of the ground state representations $L(\lambda_N)$ for large N.

12 Proof of theorem 8

To prove the theorem 8 we now fix a maximal atypical block Λ of $\mathcal{T}$ and its ground state representation $L = L(\lambda_0)$. Consider the thick triangulated subcategory $\mathcal{Z}_\Lambda$ of $\mathcal{B}$ associated to L as defined in section 8. To show that a given simple maximal atypical representation $L(\mu)$ of Λ has image in $\mathcal{Z}_\Lambda$ it suffices that it is zero in $\mathcal{B}/\mathcal{Z}_\Lambda$. If this holds for all simple objects of the block Λ, then it also holds for all objects of the block Λ.

An object A will be called a virtual ground state object, if there exists an isomorphism in $\mathcal{B}$ of the form $A \oplus A' \cong A''$ where A' and A'' is isomorphic to a finite direct sum of higher ground state objects L_N of the block Λ. We can apply the algorithms I, II and III and corollary 8 to show by induction that there exist virtual ground state objects Y and Y' and an isomorphism in $\mathcal{B}$

$$ L(\mu) \oplus Y \cong Y' \, . $$

This immediately implies that also $L(\mu)$ is a virtual ground state object.

In lemma 20 we show, that all higher ground states L_N of Λ (for all $N \geq 0$) are in $\mathcal{Z}_\Lambda$. For this we use the next

Algorithm IV. Let λ be a Kostant weight. By [BS] II, lemma 7.2 this means that the labels $\vee$ of λ define an interval $[a, .., a + n - 1]$ after the crosses have been removed. Starting from the label λ we make successively moves with the right most label $\vee$ away to the right by i steps. Let us call these new weights S^i so that S^0 is the Kostant weight we started from, and so on. Put $i = a + n - 1$. Make a first move with $[i, i + 1] = [a + n - 1, a + n]$. This move is encapsulated and gives the Loewy diagram

$$
\begin{array}{c}
S^0 \\
| \\
S^1 \\
| \\
S^0
\end{array}
$$

Next move for $[i + 1, i + 2]$ gives the Loewy diagram with four irreducible constituents

$$
\begin{array}{ccc}
 & S^1 & \\
 & | & \\
S^2 & \oplus & S^0 \\
 & | & \\
 & S^1 &
\end{array}
$$

and so one until the first move, which is not encapsulated. Here we end up in a Loewy diagram of type

$$
\begin{array}{ccccc}
 & & S^{n-1} & & \\
 & & | & & \\
S^n & \oplus & \Pi & \oplus & S^{n-2} \\
 & & | & & \\
 & & S^{n-1} & &
\end{array}
$$

with additional fifth constituent Π, where Π is of Kostant type in the given block such that compared to the Kostant weight λ we started from all labels $\vee$ have been shifted to the left by one, and hence are at the positions $[a - 1, ..., a + n - 2]$.

Lemma 20. *Suppose $L(\lambda_N) \in \mathcal{Z}_\Lambda$, then also $L(\lambda_{N+1}) \in \mathcal{Z}_\Lambda$.*

Proof. Use that $\mathcal{Z}_\Lambda$ is a thick triangulated subcategory of $\mathcal{B}$ by lemma 13. Hence it suffices that $S^0 = 0$ in $\mathcal{B}/\mathcal{Z}_\Lambda$ implies $S^i = 0$ and hence $\Pi = 0$ in $\mathcal{B}/\mathcal{Z}_\Lambda$, which obviously follows from the Loewy diagrams displayed above. Since $\Pi = L(\lambda_{N+1})$, if $S^0 = L(\lambda_N)$, we are done. QED

This shows that for every simple object $L(\mu)$ there exist A and A' in $\mathcal{Z}_\Lambda$ such that $L(\mu) \oplus A \cong A'$ in $\mathcal{B}$. Hence $L(\mu) = 0$ in $\mathcal{B}/\mathcal{Z}_\Lambda$. Therefore $L(\mu) \in \mathcal{Z}_\Lambda$. By parity reasons therefore $L(\mu) \cong m(\mu) \cdot L[p(\mu)]$ for the uniquely defined parity $p(\mu)$ (which is easily computed from the Bruhat distance from the ground state λ_0). This proves theorem 8.

13 Odds and ends

Consider a maximal atypical block of $\mathcal{T}$. Since $Ber_{m|n}$ is invertible, the tensor product with $Ber_{m|n}$ defines equivalences between maximal atypical blocks and their twisted images. Hence using a twist by a power of the Berezin we may, without restriction of generality, assume that the block Λ contains a ground state weight vector of the special form $\lambda_0 = (\lambda_1, ..., \lambda_{m-n}\ ,\ 0, ..., 0\ ;\ 0, ..., 0)$ where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{m-n} = 0$. For this see the remarks following lemma 3. Hence we may assume that the ground state is a covariant representation $L \cong \{\lambda\}$ associated to the partition $\lambda_1 + \lambda_2 + .. + \lambda_{m-n-1}$. We say that Λ is a block with normalized ground state.

Consider the K-linear triangulated tensor functor

$$\varphi : \mathcal{T}_{m|n} \longrightarrow Rep_k(H) \otimes_k \mathcal{B}_{n|n}$$

for $L = \{\lambda\} = Schur_\lambda(X)$. Since $\varphi(X) = k^{m-n} \otimes_k \mathbf{1}$ (the standard representation is not maximal atypical for $m = n$ and corollary 4), we conclude

$$\varphi(L) \cong Schur_\lambda(k^{m-n}) \otimes_k \mathbf{1}\ .$$

Since $\mathbf{1}$ is the ground state of the unique maximal atypical block of $\mathcal{T}_{n|n}$, this implies

Lemma 21. *For blocks with normalized ground states the functor φ maps ground states to ground states.*

Lemma 22. $\varphi(Ber_{m|n}) = det \otimes_k Ber_{n|n} = det \otimes \mathbf{1}[n]$.

Proof. Obvious.

14 Multiplicities

Fix a maximal atypical block Λ. Let the ground state vector of Λ be

$$\lambda = (\lambda_1, ..., \lambda_{n-m}, M, .., M; -M, ..., -M)$$

for $M = \lambda_{n-m}$. The block Λ is characterized by $(\lambda_1, .., \lambda_{m-n})$ respectively the corresponding irreducible representation $\rho = Schur_{\lambda_1,..,\lambda_{m-n}}(k^{m-n})$ in $Rep_k(H)$ with the following convention. For $M = \lambda_{m-n} < 0$ we define $Schur_{\lambda_1,..,\lambda_{m-n}}(k^{m-n}) :=$ $Schur_{\lambda_1-M,..,\lambda_{m-n}-M}(k^{m-n}) \otimes det^M$ by abuse of notation. By Lemma 22 this notation behaves nicely with respect to the triangulated tensor functor

$$\varphi : \mathcal{T}_{m|n} \longrightarrow Rep_k(H) \otimes_k \mathcal{B}_{n|n} \ .$$

Indeed, $\varphi(L(\lambda)) = \varphi(Ber^M_{m|n} \otimes Schur_{\lambda_1-M,..,\lambda_{m-n}-M}(X))$ for $X = k^{m|n}$ coincides with $det^M \otimes Schur_{\lambda_1-M,..,\lambda_{m-n}-M})(\varphi(X)) \otimes_k Ber^M_{n|n}$. Since $\varphi(X) \cong k^{m-n} \otimes_k 1$ in $\mathcal{B}$, this gives with the convention above $\varphi(L(\lambda)) = Schur_{\lambda_1,..,\lambda_{m-n}}(k^{m-n}) \otimes_k Ber^M_{n|n}$. Recall $\mathcal{B}_{n|n} = \mathcal{Z} \cong svec_K$. Using corollary 6 an obvious reexamination of the proof of theorem 8 now shows

Theorem 9. *For each weight μ in the fixed maximal atypical block Λ we have*

$$\varphi(L(\mu)) \cong m(\mu) \cdot Schur_{\lambda_1,..,\lambda_{m-n}}(k^{m-n}) \otimes_k \mathbf{1}[p(\mu)]$$

in $Rep_k(H) \otimes \mathcal{Z}$ for some integral multiplicity $m(\mu) \geq 1$, which only depends on the relative position of the weight μ with respect to the ground state weight, considered on the numberline $\mathbb{Z}$ with all crosses $\times$ removed. For the ground states the multiplicity is one.

In particular, this theorem shows that the computation of the multiplicities $m(\mu)$ can be reduced to the case $m = n$. The computation of the parity $p(\mu) = \sum_{i=1}^{n} \mu_{m+i}$ is reduced to the case of the ground state. By lemma 22 one can reduce to the case of a block with a normalized ground state, where the parity is even by the computation preceding the theorem.

The muliplicities $m(\mu)$. As already explained, as a consequence of theorem 9, we may assume $m = n$ for the computation of the multiplicities. These multiplicities are numbers attached to cup diagrams $\underline{\mu}$ with n cups (and without lines). We have already shown that the multiplicity $m(\mu)$ is one for the ground state $\mu = \mathbf{1}$. The same holds for all powers Ber^N of the Berezin by lemma 22. Hence for any completely nested cup the multiplicity $m(\mu)$ is one. To deal with a general maximal atypical weight our strategy is the following. We consider cup diagrams for various n with the aim to reduce the computation of $m(\mu)$ for a cup diagram with n cups to the case of cup diagrams with $< n$ cusps.

For a completely nested cup the multiplicity $m(\mu)$ is one. In general let μ have the sectors $S_1, .., S_r$ with length $2n_1, .., 2n_r$ with corresponding partial cup diagrams $\underline{\mu_1}, ..., \underline{\mu_r}$. Notice $n = n_1 + ... + n_r$. Each S_i defines a number interval $[a_i, b_i]$. Using the Berezin we see that the multiplicity does not change under a translation of the cup diagram. Now the algorithms II and III applied to the cup diagram of μ show, that all nested cups $\underline{\mu_i}$ can be reordered to become completely nested without destroying the sector structure of the original cup diagram μ. In this way the cup diagram can be rearranged so that all nested cups are completely nested cups. This process proves the formula

$$(*) \qquad m(\mu) \; = \; m(\nu) \cdot \prod_{i=1}^{r} m(\mu_i) \, ,$$

where ν is the cup diagram with the same sectors as μ, but so that each sector S_i defines a maximal nested cup diagram with labels $\vee$ at the position $a_i, a_i + 1, ..., a_i + n_i$.

Lemma 23. *Suppose ν is a maximal atypical weight ν with r sectors. If all sectors of the cup diagram of ν have completely nested cup diagrams of lengths say $2n_1, .., 2n_r$, then*

$$m(\nu) = \binom{n}{n_1, \ldots, n_r} \quad \textit{(multinomial coefficient)} \, .$$

Lemma 24. *For maximal atypical weights μ with n labels $\vee$ the multiplicity*

$$m(\mu) \; = \; \binom{n}{n_1, \ldots, n_r} \cdot \prod_{i=1}^{r} m(\mu_i)$$

satisfies the inequality

$$1 \leq m(\mu) \leq n! \, .$$

Equality at the right side holds if and only if the cup diagram of μ is completely unnested (i.e. all sectors have length 2). Equality on the left holds if and only if the cup diagram has only one sector which is a completely nested sector (translates of the ground state).

Proofs. Induction on n using formula (*), theorem 9 and lemma 23.

15 Proof of lemma 23

Case of one segment. Suppose ν has only one segment. Then $m(\nu) = m(n_1, .., n_r)$ depends only on the sector lengths $n_1, .., n_r$. Algorithm II applied to the first two sectors, combined with formula (*) from above, gives the recursion formula

$$2 \cdot m(u+v, n_3, .., n_r) m(u-1, 1, v-1) \; - \; m(u, v, n_3, .., n_r) \; =$$

$$m(u+v, n_3, .., n_r) m(u-1, v) m(1, v-2) \; + \; m(u+v, n_3, .., n_r) m(u, v-1) m(u-2, 1)$$

for $m(u, v, n_3, .., n_r)$ in u and v. All terms except $m(u, v, n_3, .., n_r)$ involve either fewer variables or less labels $\vee$. This allows to verify the claim by induction on the number $\sum_{i=1}^{r} n_i$ of labels $\vee$ and then of sectors r. The verification of the induction start $r = 1$ is obvious by definition. So it suffices that the multinomial coefficient $m(n_1, .., n_r) = (\sum_{i=1}^{r} n_i)! / \prod_{i=1}^{r}(n_i)!$ satisfies the recursion relation of algorithm I. The trivial property

$$m(n_1, n_2, n_3, .., n_r) \; = \; m(n_1, n_2) \cdot m(n_1 + n_2, n_3, .., n_r)$$

of multinomial coefficients allows to assume $r = 2$. The recursion formula then boils down to the identity $2uv = (u + v) + v(u - 1) + u(v - 1)$. This proves the assertion if there is only one segment.

The case of more than one segment. Now suppose ν is a maximal atypical totally nested weight with $s > 1$ segments and with a total number of r sectors of lengths $2n_1, .., 2n_r$. Notice that all segments are sectors by our assumption on ν. We then symbolically write

$$\nu \; = \; ... \; \wedge\wedge \; (S_1 \wedge ... \; q \; ... \wedge S_2) \; \wedge ... \; \text{rest with higher segments}$$

for the segment diagram, where q denotes the distance between the first and second segment. To show that the multiplicity formula of lemma 23 also holds in general we now use algorithm I to increase the size of the first sector. We assume by induction that the formula holds for maximal atypical totally nested weight with $< s$ segments or for maximal atypical totally nested weight with $\geq s$ segments and more than $2n_1$ elements in the first sector or with $\geq s$ segments and $2n_1$ elements in the first sector but smaller distance q between the first and second sector. This start of the induction is the case with one segment already considered.

First case. Suppose the distance $q = 1$.

a) Then for completely nested sectors S_1 and S_2 of length $2n_1$ and $2n_2$

$$\nu \;=\; ... \;\wedge\; (\wedge\, S_1 \wedge S_2) \wedge ... \;\;\text{rest with higher segments .}$$

b) Let $\lambda_{\vee\wedge}$ denote the weight obtained by moving the starting point of the second sector S_2 one step down so that that it touches the end of the first sector S_1. This new weight $\lambda_{\vee\wedge}$ has $s-1$ segments with segment structure

$$\lambda_{\vee\wedge} \;=\; ... \;\wedge\; (\wedge\, T_1 \wedge) \wedge ... \;\;\text{rest with higher segments}$$

whose first segment T_1 has length $2(n_1 + n_2)$ with three completely nested sectors of lengths $2n_1, 2, 2(n_2 - 1)$ respectively. Algorithm I gives three further weights:

c) The boundary move weight with segment diagram

$$... \;\wedge\; (S'_1 \wedge S'_2 \wedge) \wedge ... \;\;\text{rest with higher segments .}$$

where the first and second segments S'_1 and S'_2 are completely nested sectors of lengths $2(n_1 + 1)$ and $2(n_2 - 1)$.

d) The interval lower sector move gives a weight with $s - 1$ segments and diagram

$$... \;\wedge\; (\wedge\, T'_1 \wedge) \wedge ... \;\;\text{rest with higher segments}$$

where the segment T'_1 of length $2(n_1 + n_2)$ has two sectors. The second sector is completely nested of length $2(n_2 - 1)$. The first sector I has length $2(n_1 + 1)$ and its interior segment I^0 decomposes into two completely nested sectors of lengths $2(n_1 - 1)$ and 2 respectively.

e) The interval upper sector move gives a weight with $s - 1$ segments and diagram

$$... \;\wedge\; (\wedge\, T''_1 \wedge) \wedge ... \;\;\text{rest with higher segments}$$

where the first segment T''_1 has length $2(n_1 + n_2)$ with two sectors. The first sector is completely nested of length $2n_1$. The second sector I has length $2n_2$ and its interior segment I^0 decomposes into two completely nested sectors of lengths 2 and $2(n_1 - 2)$ respectively.

Again we show that the multinomial coefficient $m(n_1, .., n_r)$ satisfies the recursion relation of algorithm I. This suffices to prove our assertions. Again the trivial property $m(n_1, n_2, n_3, .., n_r) \;=\; m(n_1, n_2) \cdot m(n_1 + n_2, n_3, .., n_r)$ of multinomial

coefficients allows to assume $r=2$. The desired recursion equation of algorithm I, that the sum of the multiplicities of a), c), d) and e) is twice the multiplicity of b), then boils down to the binomial identity

$$2 \cdot \binom{n_1 + n_2}{n_1, 1, n_2 - 1} = \binom{n_1 + n_2}{n_1} + \binom{n_1 + n_2}{n_1 + 1} + \binom{n_1 + n_2}{n_1 + 1}\binom{n_1}{1} + \binom{n_1 + n_2}{n_1}\binom{n_2 - 1}{1}.$$

Second case. Now suppose $q \geq 2$ for the distance q between the first and the second sector.

a) Then $\nu = \ldots \wedge (S_1 \wedge \ldots q \ldots \wedge S_2) \wedge \ldots$ rest with higher segments. The first and second segments S_1, S_2 are completely nested sectors of length $2n_1$ respectively $2n_2$.

b) Consider the weight $\lambda_{\vee\wedge}$ which is obtained by moving the starting point of the second sector S_2 one step down to the left. Since $q > 1$ it does not touch the end of the first sector S_1. The new weight $\lambda_{\vee\wedge}$ still has s segments, but now with the segment structure

$$\lambda_{\vee\wedge} = \ldots \wedge (S_1 \wedge \ldots q - 1 \ldots \wedge S_2' \wedge) \wedge \ldots \text{ rest with higher segments}$$

where the second segment S_2' of lenght $2n_2$ has two completely nested sectors of lengths 2 and $2(n_2 - 1)$. The algorithm I gives two further weights:

c) If $q > 2$, the boundary move weight gives a diagram with $s + 1$ segments

$$\ldots \wedge (S_1 \wedge \ldots q - 2 \ldots \wedge T_{23} \wedge) \wedge \ldots \text{ rest with higher segments}$$

where $T_{23} = [\vee\wedge] \wedge T_2'$ has two completely nested segments $T_1' = [\vee\wedge]$ and T_2' of lengths 2 respectively $2(n_2 - 1)$.

If $q = 2$ we get a diagram with s segments

$$\ldots \wedge (S_1' \wedge T_2' \wedge) \wedge \ldots \text{ rest with higher segments}$$

where the first segment S_1' has two completely nested sectors of lengths $2n_1$ respectively 2 and the second segment T_2' is a completely nested sector of length $2(n_2 - 1)$; with distance $q = 1$ from the first sector.

e) The interval upper sector move gives a weight with s segments and diagram

$$\ldots \wedge (S_1 \wedge \ldots q - 1 \ldots \wedge T_2 \wedge) \wedge \ldots \text{ rest with higher segments}$$

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where the second segment T_2 is a sector of length $2n_2$, its interior decomposes into two completely nested sectors of lengths 2 and $2(n_2 - 2)$.

The recursion relation of algorithm II, that twice the multiplicity of b) is the sum of the multiplicities of a), c) and e), holds for the multinomial coefficient. This amounts to the binomial identity

$$2 \cdot \binom{n_1 + n_2}{n_1, 1, n_2 - 1} = \binom{n_1 + n_2}{n_1} + \binom{n_1 + n_2}{n_1}\binom{n_2 - 1}{1} + \binom{n_1 + n_2}{n_1, 1, n_2 - 1}.$$

This finally completes the proof of lemma 23 using induction on the distance q.

16 Appendix: The class ξ_n

Consider the exact BGG complex [BS2], thm. 7.3 for the Kostant weight $\mu = 0$ given by $... \to V_2 \to V_1 \to V_0 \to \mathbf{1} \to 0$ with

$$V_j = \bigoplus_{\lambda \leq 1, \; l(\lambda, 1) = j} V(\lambda) .$$

Proposition 3. *For $Gl(n|n)$ there is a nontrivial morphism $\xi_n : \mathbf{1} \to Ber_{n|n}[n]$ in $\mathcal{K}$ which becomes an isomorphism in $\mathcal{B}$. Hence Ber is contained in $\mathcal{Z}$.*

Proof. Appling the antiinvolution * we get

$$0 \to \mathbf{1} \to V_0^* \to V_1^* \to \cdots \to V_{n-1}^* \to V_n^* \to ,$$

which defines a Yoneda extension class $\xi \in Ext^n(Q, 1)$

$$0 \to \mathbf{1} \to V_0^* \to V_1^* \to \cdots \to V_{n-1}^* \to Q \to 0$$

for $Q \cong im(d^* : V_{n-1}^* \to V_n^*) \hookrightarrow V_n^*$.

Now $V_i^* = 0$ in $\mathcal{H}$, since $V([\lambda])^* = 0$ holds in $\mathcal{H}$ for all cell modules $V([\lambda]$ by the definition of $\mathcal{H}$. Hence in $\mathcal{H}$, and therefore in $\mathcal{B}$, we get

$$Q \cong \mathbf{1}[n] .$$

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We will now construct a map $i : Ber^{-1} \hookrightarrow Q$ in $\mathcal{T}$, which defines a nontrivial morphism in $\mathcal{B}$. Then ξ is nontrivial in $\mathcal{K}$, hence $i^*(\xi)$ defines a nontrivial extension in $Ext_{\mathcal{T}}^n(Ber^{-1}, 1)$.

Nontriviality. To show that a given morphism $i : Ber^{-1} \hookrightarrow Q$ is nontrivial in $\mathcal{B}$ is equivalent to show that the transposed morphism $i^* : Q^* \to Ber^{-1}$ is nontrivial in $\mathcal{B}$. We first show that $i^* : Q^* \to Ber^{-1}$ is nonzero in $\mathcal{H}$. Then i^* remains nonzero in $\mathcal{B}$, using $Q^* \cong \mathbf{1}[-n] \cong \mathbf{1}[n]$ in $\mathcal{H}$ and using that after restriction to $psl(n,n)$ the morphism i^* is in the central graded ring $R_{\mathcal{K}}^{\bullet}$ and of positive degree, hence by proposition 1 the morphism i^* can not become a zero divisor for the localization $\mathcal{B}$ and thus is a nonzero morphism in $\mathcal{B}$.

To show that $i^* \neq 0$ in $\mathcal{H}$ we argue as follows: If $i^* = 0$ in $\mathcal{H}$, then the composite morphism $V_n \twoheadrightarrow Q^* \twoheadrightarrow Ber^{-1}$, defined in $\mathcal{T}$, becomes zero in $\mathcal{H}$. Since Ber^{-1} is simple and V_n is a cell object, we can apply theorem 6 to obtain $Hom_{\mathcal{H}}(V_n, Ber^{-1}) = Hom_{\mathcal{K}}(V_n, Ber^{-1}) = Hom_{\mathcal{T}}(V_n, Ber^{-1})$. Since $V_n \twoheadrightarrow Ber^{-1}$ is an epimorphism in $\mathcal{T}$, the composite map is nonzero in $\mathcal{T}$ and therefore nonzero in $\mathcal{H}$. This completes the proof that i^* is not zero in $\mathcal{H}$. Therefore, once we have constructed an epimorphism $i^* : Q^* \to Ber^{-1}$ in $\mathcal{T}$, this proves the proposition.

Existence. To define i^* recall, that the boundary morphisms d^* are dual to the morphism d defined in [BS2]: With respect to the decomposition

$$V_n = \bigoplus_{\mu \leq 0 \, , \, l(\mu,0)=n} V(\mu) \, ,$$

the $d : V_{n-1} \to V_n$ are defined as the sum of morphisms $f_{\lambda\mu}$. See [BS2], lemma 7.1. These $f_{\lambda\mu}$ are obtained as follows: There exists a projective $P = P(\lambda)$ and an endomorphism $f : P \to P$, an filtration $N \subset M \subset P$ with $P/M = V(\lambda)$ and $M/N = V(\mu)$ such that $f(P) \subset M$ and $f(M) \subset N$ so that f induces the morphism $f_{\lambda\mu} : P/M = V(\lambda) \to M/N = V(\mu)$.

To define an epimorphism

$$i^* : Q^* \cong Im\big(d : V_n \to V_{n-1}\big) \twoheadrightarrow Ber^{-1}$$

notice that $d : V_n \to V_{n-1}$ is $\sum d_\mu$ for $d_\mu = d|_{V(\mu)} : V(\mu) \to V_{n-1}$ where $\mu < 0$ and $l(\mu,0) = n$. The cosocle of $V(\mu)$ is the simple object $L(\mu)$. Hence $V(\mu) = V(Ber^{-1})$ is the unique summand of V_n with cosocle Ber^{-1}. Since none of the morphisms d_μ is trivial in $\mathcal{T}$ (see [BS2]), the summand Ber^{-1} in the cosocle of V_n

maps nontrivially to the cosocle of its image Q^* in V_{n-1}. The only indecomposable summand $V(\mu)$ of V_n containing Ber^{-1} in its cosocle is $V(Ber^{-1})$. The cosocle of $V(Ber^{-1})$ therefore injects into Q by the definition of the morphism d. This completes the proof of proposition 3.

17 Appendix: Semisimplicity

We consider a triangulated category[1] $\mathcal{B}$, such that there are strictly full additive subcategories $\mathcal{B}_0$ and $\mathcal{B}_1$ with the following properties

1. $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$

2. $\mathcal{B}_0[1] = \mathcal{B}_1$

3. $\mathcal{B}_1[1] = \mathcal{B}_0$

4. $Hom_\mathcal{B}(\mathcal{B}_0, \mathcal{B}_1) = 0$.

By property 1. and 4. it easily follows, that the decomposition 1. is functorial such that there are adjoint functors $\tau_0 : \mathcal{B} \to \mathcal{B}_0$ and $\tau_1 : \mathcal{B} \to \mathcal{B}_1$ with functorial distinguished triangles $(\tau_0(X), X, \tau_1(X))$, s.t. $Hom_\mathcal{B}(A, X) = Hom_\mathcal{B}(A, \tau_0(X))$ for $A \in \mathcal{B}_0$ and similarly $Hom_\mathcal{B}(A, X) = Hom_\mathcal{B}(A, \tau_1(X))$ for $A \in \mathcal{B}_1$. The next lemma immediately follows from the long exact Hom-sequences attached to distinguished triangles

Extension Lemma. *If $A, C \in \mathcal{B}_0$ and (A, B, C) is a distinguished triangle, then $B \in \mathcal{B}_0$.*

Lemma 25. *$\mathcal{B}_0$ is an abelian category.*

---

[1]As pointed out by Heidersdorf, that this is a special case of a triangulated category $\mathcal{B}$ with a cluster tilting cotorsion pair $(\mathcal{U}, \mathcal{V})$ (see [Na]).

Proof. For a morphism $f : X \to Y$ in $\mathcal{B}_0$ let Z_f be a cone in $\mathcal{B}$. Put $Ker_f = \tau_0(Z_f[-1])$ and $Ker_f = \tau_0(Z_f)$. Then $Ker_f, Koker_f$ are in $\mathcal{B}_0$ and represent the kernel resp. kokernel of f in $\mathcal{B}_0$. This is an immediate consequence of the long exact Hom-sequences and property 4. Furthermore $= b \circ a$ factorizes in the form $a : X \to Z$ and $b : Z \to Y$ by the octaeder axiom, where Z is a cone of the composed morphism i defined by $Ker_f \to Z_f[-1] \to X$. By the octaeder axiom there exists a distinguished triangle $(Koker_f, Z, Y)$. Hence by the extension lemma $Z \in \mathcal{B}_0$. The octaeder axiom provides the morphisms a and b and proves $a_* : Z \cong Koker_i$ and similarly $b_* : Z \cong Ker_\pi$ for the morphism $\pi : Y \to Koker_f$. Since obviously $\mathcal{B}_0$ is an additive subcategory by the functoriality of the decomposition 1., this implies that $\mathcal{B}_0$ is an abelian category. QED

By construction the exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{B}_0$ correspond to the distinguished triangles (A, B, C) in $\mathcal{B}_0$.

Lemma 26. *The abelian category $\mathcal{B}_0$ is semisimple.*

Proof. For a short exact sequence the corresponding triangle (A, B, C) splits, since the morphism $C \to A[1]$ vanishes by property 4. QED

Corollary 9. *$\mathcal{B}$ is a semisimple abelian category.*

Suppose (X, Y, Z) is a distinguished triangle in $\mathcal{B}$. Then for $Z \in \mathcal{B}_0$ there exists an exact sequence

$$0 \longrightarrow \tau_0(X) \longrightarrow \tau_0(Y) \longrightarrow \tau_0(Z)$$

in $\mathcal{H}_0$. Similary if $X \in \mathcal{B}_0$ there exists an exacts sequence

$$\tau_0(X) \longrightarrow \tau_0(Y) \longrightarrow \tau_0(Z) \longrightarrow 0 .$$

These statements follow immediately from the long exact Hom-sequences and the assumptions 2. and 3. Using the argument of [KW], thm. 4.4 this implies

Lemma 27. . *Put $H^i(X) = \tau_0(X[i])$. Then for a distinguished triangle (X, Y, Z) in $\mathcal{B}$ there exists a long exact cohomology sequence in $\mathcal{B}_0$*

$$... \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to ... .$$

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In our case $\mathcal{B} = \mathcal{B}_{m|n}$ satisfies these properties 1.-4., and indeed the suspension functor is induced by the parity shift functor Π on $svec_K$ via the equivalence $\mathcal{B} \sim Rep_k(H) \otimes_k svec_K$ of categories. Using $A[2] \cong A$ defined by $id_A \otimes \zeta_2^{-1}$ and the identification $\mathcal{B} = sRep_K(H)$, we may identify the suspension functor with the parity shift functor, with functorial isomorphisms $H^{2i}(A) \cong A_+$ and $H^{2i+1}(A) \cong A_-$ for $A = A_+ \oplus A_-$ in $sRep_K(H)$. Hence the long exact sequence of the cohomology becomes a hexagon. $\mathcal{B}_{m|n}$ is a Π-category, a triangulated category enhanced by a super space structure, in the following sense:

By definition a Π-category is a triangulated category with the properties 1.-4. from above such that there exist functorial isomorphisms $\Pi^2(A) \cong A$ for the suspension $\Pi(A) := A[1]$. For a functor $F : \mathcal{A} \to \mathcal{B}$ then notice $F(A) \cong F(A)_+ \oplus F(A)_-$, where $H^0 F(A) = F(A)_+$ and $H^1 F(A) = F(A)_-$. An additive functor F from an abelian category $\mathcal{A}$ to a Π-category $\mathcal{B}$ will be called weakly exact, if $F(A)_+$ and $F(A)_-$ transform short exact sequences $0 \to A \to B \to C \to 0$ into exact hexagons in $\mathcal{B}_0$

$$
\begin{array}{ccc}
 & F(B)_+ \longrightarrow F(C)_+ & \\
\nearrow & & \searrow \\
F(A)_+ & & F(A)_- \\
\nwarrow & & \swarrow \\
 & F(C)_- \longleftarrow F(B)_- & \\
\end{array}
$$

From the definition of the functor $\varphi = \gamma \circ \beta \circ \alpha$ the following then is obvious

Lemma 28. *The functor* $\varphi : \mathcal{T}_{m|n} \to sRep_K(H)$ *is weakly exact.*

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